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Holographic description of AdS cosmologies

Thomas Hertog and Gary T. Horowitz

Department of Physics, UCSB
Santa Barbara, CA 93106, U.S.A.
E-mail: Hertog@vulcan.physics.ucsb.edu, gary@physics.ucsb.edu

Abstract: To gain insight in the quantum nature of the big bang, we study the dual field theory description of asymptotically anti-de Sitter solutions of supergravity that have cosmological singularities. The dual theories do not appear to have a stable ground state. One regularization of the theory causes the cosmological singularities in the bulk to turn into giant black holes with scalar hair. We interpret these hairy black holes in the dual field theory and use them to compute a finite temperature effective potential. In our study of the field theory evolution, we find no evidence for a "bounce" from a big crunch to a big bang. Instead, it appears that the big bang is a rare fluctuation from a generic equilibrium quantum gravity state.

Keywords: AdS-CFT and dS-CFT Correspondence, Black Holes in String Theory, Cosmology of Theories beyond the SM.

Contents

1. Introduction

One of the main goals of quantum gravity is to provide a better understanding of the big bang singularity in cosmology. This is essential for cosmology to be a truly predictive science, that explains how the distinctive features of the universe emerged from the early quantum gravitational phase and why they are what they are. A long-standing issue is whether cosmological singularities represent a true beginning or end of evolution. More generally, one would like to understand how semiclassical spacetime and our usual notion of time arise from the past singularity. Various suggestions have been made for how this quantum gravity transition happens. These include the no boundary wave function [1] that describes creation ex nihilo, and the chaotic initial conditions proposed in [2]. Alternatively, it is possible that evolution essentially continues through the singularity, with an immediate transition from a big crunch to a big bang [3, 4].

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Since our usual notions of space and time are likely to break down near cosmological singularities, a more promising approach to study the problem of initial conditions in cosmology is to find a dual description in terms of more fundamental variables. In string theory we do not yet have a dual description of real cosmologies, but we do have the celebrated AdS/CFT correspondence [5] which provides a non-perturbative definition of string theory on asymptotically anti-de Sitter (AdS) spacetimes in terms of a conformal field theory (CFT). We have recently constructed examples of solutions in $\mathcal{N} = 8$, $D = 4$ supergravity where smooth, asymptotically AdS initial data emerge from a big bang in the past and evolve to a big crunch singularity in the future [6]. The AdS/CFT duality should provide a precise framework in which the quantum nature of the cosmological singularities can be understood.[1]

With standard AdS-invariant boundary conditions in the bulk, the dual CFT is the usual $2 + 1$ theory on a stack of M2-branes. To construct the AdS cosmologies, we have generalized the boundary conditions on one of the tachyonic scalars in the theory (while preserving the AdS symmetries), which corresponds to modifying the CFT by a triple trace operator. In this paper we study the dual field theory to gain insight in the quantum nature of the cosmological singularities.

This field theory appears to have a potential unbounded from below, which makes it difficult to analyze. However the fact that one can map the problem of cosmological singularities in quantum gravity to a problem in an ordinary nongravitational field theory appears to be a significant advance. We use a variety of approaches to study this dual theory. One regularization of the unbounded potential turns the cosmological singularity into a large black hole with scalar hair.[2] We interpret these hairy black holes in the dual field theory and use them to compute a finite temperature effective potential.

While a complete quantum understanding of cosmological singularities is still not available, we are led to a picture of the big bang as a rare fluctuation in a typical quantum gravity state in which all the Planck scale degrees of freedom are excited. This could help explain the origin of the second law of thermodynamics. Penrose [10] has long argued that past singularities such as the big bang must be very different from future singularities like a big crunch, since the former must correspond to a state of very low entropy. We will indeed find such an asymmetry between past and future singularities. Whether the entropy is low enough to explain observations will require a more detailed cosmological model and in particular, a better understanding how semiclassical spacetime emerges from a generic quantum gravity state.

A brief outline of this paper is as follows. In the next section we review the construction of asymptotically anti de Sitter initial data which evolves to a big crunch. In section 3, we discuss the dual field theory evolution using several different approximations. A regularization of the field theory is introduced in section 4 and the connection with hairy

[1] A different approach to holographic cosmology has been discussed in [7].

[2] The hair comes from one of the tachyonic scalars in the theory. It has been shown [8] there are no hairy AdS black holes where the scalar field asymptotically tends to the true minimum of its potential. The first examples of AdS black holes with (non-tachyonic) scalar hair were given in [9].

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black holes is explored. This leads to the picture of the big bang as a rare fluctuation. The final section contains some directions for further work. In the course of our discussion, we introduce several functions of the form $\beta(\alpha)$. To avoid confusion, we list all of them in an appendix, together with their definition.

2. Anti-de Sitter cosmology

2.1 Setup

We consider the low energy limit of M theory with $AdS_4 \times S^7$ boundary conditions. The massless sector of the compactification of $D = 11$ supergravity on S^7 is $\mathcal{N} = 8$ gauged supergravity in four dimensions [11]. The bosonic part of this theory involves the graviton, 28 gauge bosons in the adjoint of SO(8), and 70 real scalars, and admits AdS_4 as a vacuum solution. It is possible to consistently truncate this theory to include only gravity and a single scalar with action [12]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 + 2 + \cosh(\sqrt{2}\phi)\right] \tag{2.1}$$

where we have set $8\pi G = 1$ and chosen the gauge coupling so that the AdS radius is one. The potential has a maximum at $\phi = 0$ corresponding to an AdS_4 solution with unit radius. It is unbounded from below, but small fluctuations have $m^2 = -2$, which is above the Breitenlohner-Freedman bound $m^2_{BF} = -9/4$ [13], so with the usual boundary conditions AdS_4 is stable [14, 15].

2.2 Boundary conditions

We will work in global coordinates in which the AdS_4 metric takes the form

$$ds_0^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_2 \,. \tag{2.2}$$

In all asymptotically AdS solutions, the scalar ϕ decays at large radius as

$$\phi(r) = \frac{\alpha}{r} + \frac{\beta}{r^2} \tag{2.3}$$

where α and β can depend on the other coordinates. The standard boundary conditions correspond to either $\alpha = 0$ or $\beta = 0$ [13, 16]. It was shown in [17] that $\beta_k = -k\alpha^2$ (with k an arbitrary constant) is another possible boundary condition that preserves all the asymptotic AdS symmetries. One can consider even more general boundary conditions $\beta = \beta(\alpha)$. Although these will generically break some of the asymptotic AdS symmetries, they are invariant under global time translations. Hence there is still a conserved total energy, as we now show.

As discussed in [17], the usual definition of energy in AdS diverges whenever $\alpha \neq 0$. This is because the backreaction of the scalar field causes certain metric components to fall off slower than usual. In particular, one has [17]

$$\begin{aligned} g_{rr} &= \frac{1}{r^2} - \frac{(1+\alpha^2/2)}{r^4} + O\left(\frac{1}{r^5}\right) & g_{tt} &= -r^2 - 1 + O\left(\frac{1}{r}\right) \\ g_{tr} &= O\left(\frac{1}{r^2}\right) & g_{ab} &= \bar{g}_{ab} + O\left(\frac{1}{r}\right) \\ g_{ra} &= O\left(\frac{1}{r^2}\right) & g_{ta} &= O\left(\frac{1}{r}\right). \end{aligned} \tag{2.4}$$

The expression for the conserved mass depends on the asymptotic behavior of the fields and is defined as follows. Let ξ^μ be a timelike vector which asymptotically approaches a (global) time translation in AdS. The hamiltonian takes the form

$$H = \int_\Sigma \xi^\mu C_\mu + \text{surface terms} \tag{2.5}$$

where Σ is a spacelike surface, C_μ are the usual constraints, and the surface terms should be chosen so that the variation of the hamiltonian is well defined. The variation of the usual gravitational surface term is given by

$$\delta Q_G[\xi] = \frac{1}{2}\oint dS_i \bar{G}^{ijkl}(\xi^\perp \bar{D}_j \delta h_{kl} - \delta h_{kl}\bar{D}_j \xi^\perp) \tag{2.6}$$

where $G^{ijkl} = \frac{1}{2}g^{1/2}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl})$, $h_{ij} = g_{ij} - \bar{g}_{ij}$ is the deviation from the spatial metric $\bar{g}_{ij}$ of pure AdS, $\bar{D}_i$ denotes covariant differentiation with respect to $\bar{g}_{ij}$ and $\xi^\perp = \xi \cdot n$ with n the unit normal to Σ. Since our scalar field is falling off more slowly than usual if $\alpha \neq 0$, there is an additional scalar contribution to the surface terms. Its variation is simply

$$\delta Q_\phi[\xi] = -\oint \xi^\perp \delta\phi D_i \phi dS^i\,. \tag{2.7}$$

Using the asymptotic behavior (2.3) this becomes

$$\delta Q_\phi[\xi] = r\oint (\alpha\delta\alpha) d\Omega + \oint [\delta(\alpha\beta) + \beta\delta\alpha] d\Omega\,. \tag{2.8}$$

Since there is a term proportional to the radius of the sphere, this scalar surface term diverges. However, this divergence is exactly canceled by the divergence of the usual gravitational surface term (2.6). The total charge can therefore be integrated, yielding

$$Q[\xi] = Q_G[\xi] + r\oint \frac{\alpha^2}{2} d\Omega + \oint [\alpha\beta + W(\alpha)] d\Omega \tag{2.9}$$

where we have defined

$$W(\alpha) = \int_0^\alpha \beta(\tilde{\alpha}) d\tilde{\alpha}\,. \tag{2.10}$$

In addition to canceling the divergence in (2.9), the gravitational surface term contributes a finite amount M_0. For the spherically symmetric solutions we consider below, this is just the coefficient of the $1/r^5$ term in g_{rr}. Since α and β are now independent of angles, the total mass becomes [18]

$$M = 4\pi(M_0 + \alpha\beta + W)\,. \tag{2.11}$$

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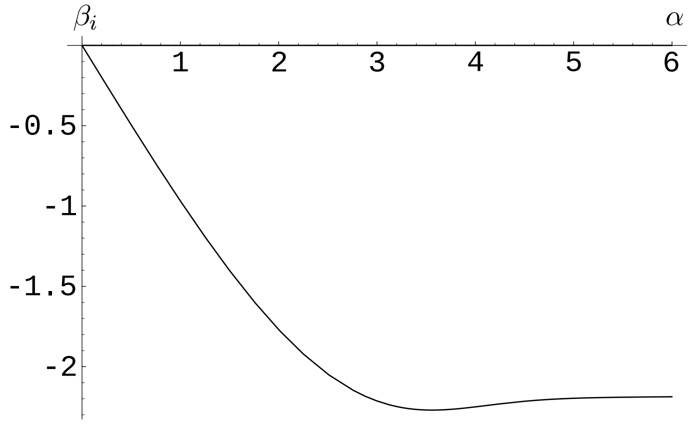

Figure 1: The function β_i obtained from the instantons.

2.3 Big bang/big crunch AdS cosmologies

We now review the big bang/big crunch AdS cosmologies of [6] that are solutions of (2.1) with boundary conditions

$$\beta_k = -k\alpha^2 \tag{2.12}$$

on the scalar field. One first finds an $O(4)$-invariant euclidean instanton solution of the form

$$ds^2 = \frac{d\rho^2}{b^2(\rho)} + \rho^2 d\Omega_3 \tag{2.13}$$

and $\phi = \phi(\rho)$. The field equations determine b in terms of ϕ

$$b^2(\rho) = \frac{2V\rho^2 - 6}{\rho^2\phi'^2 - 6} \tag{2.14}$$

and the scalar field ϕ itself obeys

$$b^2\phi'' + \left(\frac{3b^2}{\rho} + bb'\right)\phi' - V_{,\phi} = 0 \tag{2.15}$$

where prime denotes ∂_ρ.

Regularity at the origin requires $\phi'(0) = 0$. Thus the instanton solutions can be labeled by the value of ϕ at the origin. For each $\phi(0)$, one can integrate (2.15) and get an instanton. Asymptotically one finds $\phi(\rho) = \alpha/\rho + \beta/\rho^2$, where α and β are now constants. Hence for each $\phi(0)$ one obtains a point in the (α, β) plane. Repeating for all $\phi(0)$ yields a curve $\beta_i(\alpha)$ where the subscript indicates this is associated with instantons. This curve is plotted in figure 1. (Since the potential $V(\phi)$ is even, it suffices to consider positive $\phi(0)$ which corresponds to positive α.)

The slice through the instanton obtained by restricting to the equator of the S^3 defines time symmetric initial data for a lorentzian solution. The euclidean radial distance ρ simply becomes the radial distance r in this initial data. So given a choice of boundary condition $\beta(\alpha)$, one can obtain suitable initial data by first selecting the instanton corresponding to a point where the curve $\beta_i(\alpha)$ intersects $\beta(\alpha)$, and then taking a slice through this instanton.

Clearly all AdS-invariant boundary conditions (2.12) admit precisely one instanton solution. Furthermore, the mass (2.11) of the resulting initial data is given by

$$M = 4\pi \left(M_0 - \frac{4}{3} k\alpha^3 \right) . \tag{2.16}$$

From the asymptotic form of (2.14) it follows that $M_0 = 4k\alpha^3/3$. Thus this initial data corresponds to a zero mass solution, consistent with its interpretation as the solution AdS_4 decays into.[3]

With $\beta_k = -k\alpha^2$ boundary conditions, the evolution of the initial data defined by the instanton geometry is simply obtained by analytic continuation [19]. The origin of the euclidean instanton becomes the lightcone of the lorentzian solution. Outside the lightcone, the solution is given by (2.13) with $d\Omega_3$ replaced by three dimensional de Sitter space. The scalar field ϕ remains bounded in this region. On the light cone we have $\phi = \phi(0)$ and $\partial_t \phi = 0$ (since $\phi_{,\rho} = 0$ at the origin in the instanton). Inside the lightcone, the SO$(3,1)$ symmetry ensures that the solution evolves like an open FRW universe,

$$ds^2 = -dt^2 + a^2(t) d\sigma_3 \tag{2.17}$$

where $d\sigma_3$ is the metric on the three dimensional unit hyperboloid. Under evolution ϕ rolls down the negative potential. This causes the scale factor $a(t)$ to vanish in finite time, producing a big crunch singularity.

To verify that this analytic continuation indeed satisfies our boundary condition, we must do a coordinate transformation in the asymptotic region outside the light cone. The relation between the usual static coordinates (2.2) for AdS_4 and the SO$(3,1)$ invariant coordinates,

$$ds^2 = \frac{d\rho^2}{1+\rho^2} + \rho^2(-d\tau^2 + \cosh^2 \tau d\Omega_2) \tag{2.18}$$

is

$$\rho^2 = r^2 \cos^2 t - \sin^2 t . \tag{2.19}$$

Hence the asymptotic behavior of ϕ in global coordinates is given by

$$\phi(r) = \frac{\tilde{\alpha}}{r} - \frac{k\tilde{\alpha}^2}{r^2} + O(r^{-3}) \tag{2.20}$$

where $\tilde{\alpha} = \alpha / \cos t$. This clearly satisfies the boundary condition (2.12), but $\tilde{\alpha}$ is now time dependent and blows up as $t \to \pi/2$, when the singularity hits the boundary.

For the purpose of understanding cosmological singularities in M theory, one can forget the origin of this solution as the analytic continuation of an instanton. We have simply found an explicit example of asymptotically AdS initial data which evolves to a big crunch. Since the initial data is time symmetric, there is also a big bang in the past (see figure 2).

[3] One can also show that this instanton has finite action [6] which is large for small k and small for large k.

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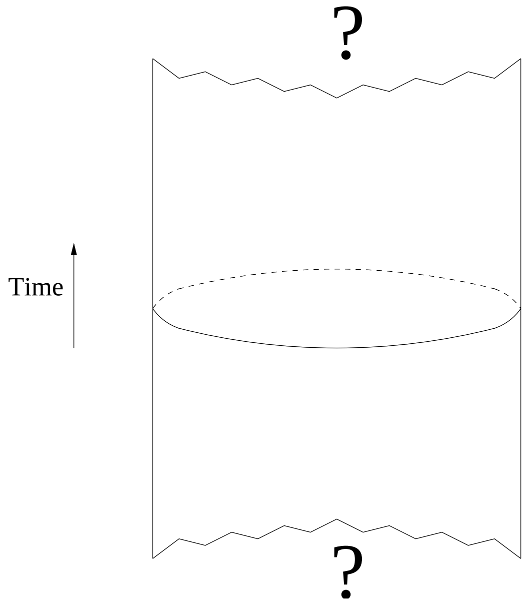

Figure 2: Anti-de Sitter cosmologies.

This theory also has static spherically symmetric solitons [18]. Another family of asymptotically AdS cosmologies can be found by starting with the profile $\phi_s(r)$ of one of these solitons. Rescaling the soliton configuration ϕ_s to $\phi_\lambda(r) = \phi_s(\lambda r)$ with sufficiently small λ gives negative mass initial data for time-dependent solutions [6]. Since for small λ we have a large central region where ϕ is essentially constant and away from the maximum of the potential, it follows that the field must evolve to a spacelike singularity. The singularity that develops cannot be hidden behind an event horizon, because all spherically symmetric, asymptotically AdS black holes have positive mass [17]. Instead, one expects it to continue to spread, cutting off all space. This construction also leads to singularities in both the past and future. In addition, there should be solutions with only one singularity, e.g., a big crunch in the future which approach a soliton in the past.

3. Dual field theory evolution

3.1 Dual CFT

We now turn to the dual field theory interpretation. M theory on spacetimes which asymptotically approach $AdS_4 \times S^7$ is dual to the 2+1 conformal field theory (CFT) describing the low energy excitations of a stack of N M2-branes. In this correspondence, scalar modes with $\beta = 0$ boundary conditions correspond to physical states, and adding nonzero β corresponds to modifying the CFT action. Our bulk scalar ϕ is dual to a dimension one operator $\mathcal{O}$. One way of obtaining this CFT is by starting with the field theory on a stack of D2-branes and taking the infrared limit. In that description [20],

$$\mathcal{O} = \frac{1}{N} Tr T_{ij} \varphi^i \varphi^j \tag{3.1}$$

where T_{ij} is symmetric and traceless and φ^i are the adjoint scalars.

Note that physical states are associated with modes with the slower fall-off. This is possible since the mass is tachyonic and close to the Breitenlohner-Freedman bound. The field theory dual to the "standard" quantization, where physical states are described by modes with $\phi = \beta/r^2$ asymptotically, can be obtained by adding the double trace term $\frac{f}{2}\int \mathcal{O}^2$ to the action [21, 22]. This is a relevant perturbation and the infrared limit is another CFT in which $\mathcal{O}$ has dimension two. As described in [17], the AdS invariant boundary conditions $\beta_k = -k\alpha^2$ correspond instead to adding a triple trace term to the action

$$S = S_0 - \frac{k}{3}\int \mathcal{O}^3 \,. \tag{3.2}$$

The extra term has dimension three, and hence is marginal and preserves conformal invariance, at least to leading order in $1/N$. In general, imposing nontrivial boundary conditions $\beta(\alpha)$ in the bulk corresponds to adding a multi-trace interaction $\int W(\mathcal{O})$ to the CFT action, such that after formally replacing $\mathcal{O}$ by its expectation value α one has [21, 23]

$$\beta = \frac{\delta W}{\delta\alpha} \,. \tag{3.3}$$

3.2 Semiclassical field theory evolution

We have seen that the coefficient $\tilde{\alpha}$ of the bulk solution (2.20) diverges as $t \to \pi/2$, when the big crunch singularity hits the boundary. Since the coefficient of $1/r$ is interpreted as the expectation value of $\mathcal{O}$ in the dual CFT, this shows that to leading order in $1/N$, $\langle\mathcal{O}\rangle$ diverges in finite time.

A qualitative explanation for this field theory behavior is the following. The term we have added to the action is not positive definite. Since the energy associated with the asymptotic time translation in the bulk can be negative [17] (and is in fact unbounded from below[4]), the dual field theory should also admit negative energy states and have a spectrum unbounded from below. This shows that the usual vacuum must be unstable, and that there are (nongravitational) instantons which describe its decay. After the tunneling, the field rolls down the potential and becomes infinite in finite time.

A semiclassical analysis supports this reasoning. If we neglect for a moment the nonabelian structure and identify $\mathcal{O}$ with φ^2, we are led to consider a single scalar field theory with standard kinetic term and potential,

$$V = \frac{1}{8}\varphi^2 - \frac{k}{3}\varphi^6 \tag{3.4}$$

where the quadratic term corresponds to the conformal coupling of φ to the curvature of S^2,[5] and the second term represents the second term in (3.2). Although this is clearly a huge simplification of the field theory, at the classical level it captures the bulk behavior in

[4]One can obtain solutions with arbitrarily negative energy by taking initial data for a static soliton and rescaling the radial variable. The rescaling does not change our boundary condition (2.12).

[5]Since we have set the AdS radius equal to one, the dual field theory lives on $S^2 \times R$ where the sphere also has unit radius.

a surprisingly quantitative way. In particular, it admits the following exact homogeneous classical solution,

$$\varphi(t) = \frac{C}{\cos^{1/2} t} \tag{3.5}$$

where $C = (3/8k)^{1/4}$. This solution has zero energy since the field starts at rest where the potential vanishes. Hence it is analogous to the solution obtained by analytically continuing the instanton. Since φ^2 is identified with $\tilde{\alpha}$ on the bulk side, the time dependence of this solution agrees with that predicted from supergravity, including the fact that the field diverges at $t \to \pi/2$. Furthermore, from figure 1 it follows that for large α, i.e. large $\phi(0)$, we have $\alpha \sim k^{-1/2}$, since β tends to a constant. Remarkably, this scaling is also reproduced by the field theory solution above.

In this model field theory, the usual vacuum at $\varphi = 0$ is perturbatively stable but nonperturbatively unstable. There are (nongravitational) instantons which describe the semiclassical decay of the usual vacuum. For small k, the potential barrier is large, and the instanton action is large. So tunneling is suppressed. For large k, the barrier is small and tunneling is not suppressed. (This agrees with the action of the gravitational instantons in the bulk.) After the tunneling, the field rolls down the potential and becomes infinite in finite time. So a semiclassical analysis suggests that the CFT does not have well defined evolution for all time.

3.3 Quantum mechanics

We have seen that evolution ends in finite time in the semiclassical description of a simplified version of the dual field theory. This agrees with the supergravity result. This conclusion changes dramatically, however, if one considers the quantum mechanics of the potential (3.4). That is, we again concentrate on the homogeneous mode $\varphi(t) = x(t)$ only, but now treat it quantum mechanically. The quantum mechanics of unbounded potentials of this type is well understood and discussed in detail in [24, 25].

Like the classical trajectories, a wave packet with an energy distribution peaked at some value E that moves in a potential of the form $-kx^6$ (with $k > 0$) will reach infinity in finite time. Thus a packet can 'disappear' and probability is apparently lost. But in quantum mechanics this problem can be dealt with by constructing a self-adjoint extension of the hamiltonian $H = -(1/2)(d^2/dx^2) + V(x)$, which is done by carefully specifying its domain. Once a domain is chosen, the hamiltonian is self-adjoint and unitary time evolution is guaranteed.

One can construct these domains by finding the corresponding set of eigenfunctions. At large x the WKB approximation is accurate and the wave functions for each energy $E > 0$ are

$$\chi_E^{\pm}(x) = \left(2E + 2kx^6\right)^{-1/4} e^{\pm i \int_0^x \sqrt{2E+2ky^6}dy} . \tag{3.6}$$

The hamiltonian is not hermitean on a domain containing all wave functions of this form. For instance, if $\phi_1 = \chi_E^+$ and $\phi_2 = \chi_{E'}^+$ or $\chi_{E'}^-$ then $(H\phi_1, \phi_2) \neq (\phi_1, H\phi_2)$. It can be made hermitean, however, by selecting a particular linear combination of χ_E^+ or χ_E^-. Let (again at large x)

$$\psi_E^\gamma = \left(2E + 2kx^6\right)^{-1/4} \cos\left(\int_0^x \sqrt{2E + 2ky^6}dy + \omega_E^\gamma\right) \tag{3.7}$$

where

$$\omega_E^\gamma = \gamma - \int_0^\infty \left(\sqrt{2E + 2ky^6} - \sqrt{2ky^6}\right) dy \tag{3.8}$$

and γ is an arbitrary phase. This choice of ω_E^γ makes ψ_E^γ approach a real, energy-independent function $\bar{\psi}^\gamma$ at large x. The complete set ψ_E^γ, for fixed γ, can be used to define the domain of H. The hamiltonian defined on this domain is hermitean, since all wave functions in the domain approach a (complex) constant times the same real function of x. Notice, however, that since γ is arbitrary, there is a one-parameter family of self-adjoint hamiltonians, each of which results in a different unitary time evolution. Which self-adjoint extension is chosen by string theory (if any) is an interesting open question. Our point here is only to show that the quantum mechanical description indicates that the big crunch is not an endpoint of evolution.

The fact that H is self-adjoint means that probability does not leave the system. Yet the center of a wave packet follows essentially the classical trajectory and still reaches infinity in finite time. What happens [25] is that a right-moving wave packet bounces off infinity and reappears as a left-moving wave packet. Hence the quantum mechanics of the potential (3.4) in the dual field description of our AdS cosmologies indicates that the big crunch is not an endpoint of evolution. Furthermore, it shows that for exactly homogeneous initial data there is a bounce through the singularity, as envisioned in the pre-big bang [3] and cyclic universe [4] models.

There is another approach to applying quantum mechanics to potentials unbounded from below based on a PT symmetry [26]. However this approach is motivated by analytic continuation from the harmonic oscillator and results in a positive spectrum. In contrast, for all self adjoint extensions described above, there is an infinite (discrete) set of negative energies. Since the bulk theory is known to have negative energy solutions, the approach using self adjoint extensions seems more appropriate.

3.4 Full quantum field theory

We have seen that a quantum mechanical analysis of the homogeneous mode of the dual field indicates that evolution is unitary and that there is an immediate transition from a big crunch to a big bang. It is natural to ask if one should expect this conclusion to hold also in the full field theory. It is not known if self-adjoint extensions can be constructed in field theories with potentials of this form. However, even if one can define a unitary evolution, the full field theory evolution is likely to be very different, because our discussion of the quantum mechanics obviously neglects the possibility of particle creation.

It is well known that a scalar field that oscillates near the minimum of its effective potential rapidly converts its energy into particles that are produced during these oscillations. It was recently found, however, that in many theories where the scalar field rolls down from the top of its effective potential towards the true minimum, particles are produced in great numbers while the field is rolling down. This phenomenon is called tachyonic preheating [27, 28]. It happens essentially because the effective negative mass term in the potential causes long wavelength quantum fluctuations to grow exponentially.

Tachyonic preheating is so efficient that in many theories most of the initial potential energy density is converted into the energy of scalar particles well before the field reaches the true minimum. Thus a prolonged stage of oscillations of the homogeneous component of the scalar field around the true minimum of the potential does not exist in spontaneous symmetry breaking.

Tachyonic preheating occurs, therefore, also in the dual field theory description of our AdS cosmologies, where the supergravity initial data correspond to a homogeneous field theory configuration high up the potential. This means that even if the field theory has well-defined evolution for all time, it will in general not be dual to a bounce through the singularity. Indeed this would require the miraculous conversion of all the energy back into the homogeneous mode!

As an aside, we note that there is a long history of studying cosmological singularities via a "minisuperspace approximation", in which one first restricts to homogeneous cosmologies and then quantizes the remaining finite number of degrees of freedom. It is clear from the above discussion that this approach misses a key physical effect.[6] The possibility of exciting all the inhomogeneous modes can dramatically change the evolution near a singularity. It is amusing to recall that before the discovery of the singularity theorems, it was widely believed that (classical) singularities would only arise in highly symmetric collapse, and generic collapse would be nonsingular. We now see that the situation in quantum gravity is almost the opposite: a strictly homogeneous collapse results in a simple bounce, while a generic collapse does not.

4. From black holes to big crunch

4.1 Regularized field theory

The main difficulty in analyzing our dual field theory is clearly that the potential is unbounded from below. If there was a stable ground state, a homogeneous field rolling down the steep potential would evolve to a thermal state about the global minimum of the potential. One can regulate our potential and produce a stable ground state by, e.g., adding to the CFT lagrangian

$$W(\mathcal{O}) = -\frac{k}{3}\int \mathcal{O}^3 + \frac{\epsilon}{4}\int \mathcal{O}^4 \,. \tag{4.1}$$

Neglecting again the nonabelian structure this gives the following potential

$$V = \frac{1}{8}\varphi^2 - \frac{k}{3}\varphi^6 + \frac{\epsilon}{4}\varphi^8 \,. \tag{4.2}$$

Provided ϵ is sufficiently small, this has a local minimum at $\varphi = 0$ and a global minimum at $\varphi^2 \sim k/\epsilon$.

[6] See [29] for an earlier objection to minisuperspace.

The operator $\mathcal{O}^4$ is not renormalizable, but to leading order in $1/N$ this does not cause any difficulty. Its only effect in the bulk is to modify the scalar boundary condition which according to (3.3) becomes

$$\beta_{k,\epsilon} = -k\alpha^2 + \epsilon\alpha^3 \,. \tag{4.3}$$

This modification will slightly change our initial data and can affect its evolution. However, its effect on evolution can only be appreciable in the "corners" of figure 2 where the singularity hits the boundary at infinity. This is because the change in the boundary conditions is negligible until α becomes large, and causality then restricts its effect to these regions. In particular, since the evolution of the $\epsilon = 0$ data has trapped surfaces, a singularity will still form in the central region. However it is now possible that the singularity is enclosed inside a large black hole and does not extend out to infinity. We now investigate this possibility.

4.2 Black holes with scalar hair

Appropriate initial data for the modified boundary conditions (4.3) is obtained by taking an instanton with $\beta_i(\alpha) = \beta_{k,\epsilon}(\alpha)$ and slicing it through the center. From figure 1, it is clear that there are two possible instantons satisfying this condition. We are interested in the one with $\beta \approx -k\alpha^2$ so the initial data is close to the one we studied in section 2. The evolution of this initial data cannot be obtained by analytic continuation of the instanton since that solution will not satisfy the new boundary conditions. A full description of the evolution probably requires numerical relativity. However, we can at least ask if there is a static black hole with the right mass to form from our initial data. The instanton field equation (2.14) always yields $M_0 = -4\alpha\beta/3$. So from (2.11), the mass of our new initial data is slightly negative[7]

$$M = 4\pi\left[-\frac{4}{3}\alpha\beta_{k,\epsilon} + \alpha\beta_{k,\epsilon} - \frac{1}{3}k\alpha^3 + \frac{1}{4}\epsilon\alpha^4\right] = -\frac{\pi}{3}\epsilon\alpha^4 \,. \tag{4.4}$$

Since the energy is conserved during evolution, a black hole can form only if there exist negative mass black holes. These black holes will necessarily have scalar hair, so we have to study hairy black holes with boundary conditions (4.3).

It was shown in [17] that the theory (2.1) with AdS-invariant boundary conditions (2.12) admits black hole solutions with scalar hair.[8] For a given choice of $k \neq 0$, there is a one-parameter family of hairy black holes. The solutions can be characterized by their conserved mass (2.16), which uniquely determines the horizon radius R_e as well as the value ϕ_e of the scalar field at the horizon of this class of solutions. The hairy black holes provide an example of black hole non-uniqueness, however, since Schwarschild-AdS is also a solution for all boundary conditions. We will discuss below how the dual field theory description resolves this non-uniqueness.

[7]The theory with modified boundary conditions has negative mass solutions, but it can be shown there is a lower bound on the mass and hence a stable ground state [32], as suggested by the dual field theory potential (4.2).

[8]See [30, 31] for other examples of black hole solutions with tachyonic scalar hair.

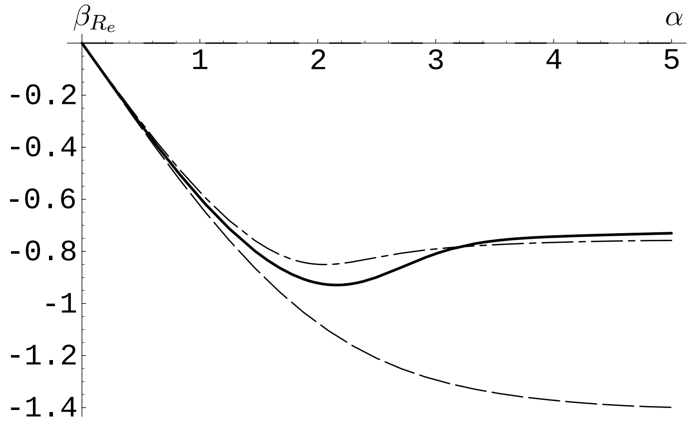

Figure 3: The functions $\beta(\alpha)$ obtained from the solitons and from hairy black holes of two different sizes. The full line shows the soliton curve $\beta_s(\alpha)$, the dot-dashed line shows the $\beta_{R_e}(\alpha)$ curve for $R_e = .2$ black holes and the dashed line is the $R_e = 1$ curve.

The hairy black holes all have positive mass (2.16) in this theory [17], so the zero mass initial data defined by the instantons cannot evolve to a black hole. Indeed we have seen that with AdS-invariant boundary conditions, initial data of this type produce a big crunch. This is not necessarily the case, however, if one evolves with the modified boundary conditions (4.3). We now show that with boundary conditions $\beta_{k,\epsilon}$, the bulk theory admits a second branch of regular black hole solutions with scalar hair, which includes negative mass black holes.

We find the hairy black holes by numerically integrating the field equations for static, spherically symmetric solutions. Writing the metric as

$$ds^2 = -h(r)e^{-2\delta(r)}dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega_2 \tag{4.5}$$

the Einstein equations read

$$h\phi_{,rr} + \left(\frac{2h}{r} + \frac{r}{2}\phi_{,r}^2 h + h_{,r}\right)\phi_{,r} = V_{,\phi} \tag{4.6}$$

$$1 - h - rh_{,r} - \frac{r^2}{2}\phi_{,r}^2 h = r^2 V(\phi) \tag{4.7}$$

$$\delta_{,r} = -\frac{1}{2} r\phi_{,r}^2 \,. \tag{4.8}$$

We integrate the field equations outward from the horizon. Regularity at the event horizon R_e requires

$$\phi_{,r}(R_e) = \frac{R_e V_{,\phi}(\phi_e)}{1 - R_e^2 V(\phi_e)} \,. \tag{4.9}$$

The scalar field asymptotically behaves as (2.3), so we obtain a point in the (α, β) plane for each combination (R_e, ϕ_e). Repeating for all ϕ_e gives a curve $\beta_{R_e}(\alpha)$. In figure 3 we show this curve for hairy black holes of two different sizes. We also show the curve obtained in a similar way for regular solitons, which were discussed in [18]. As one increases R_e, the

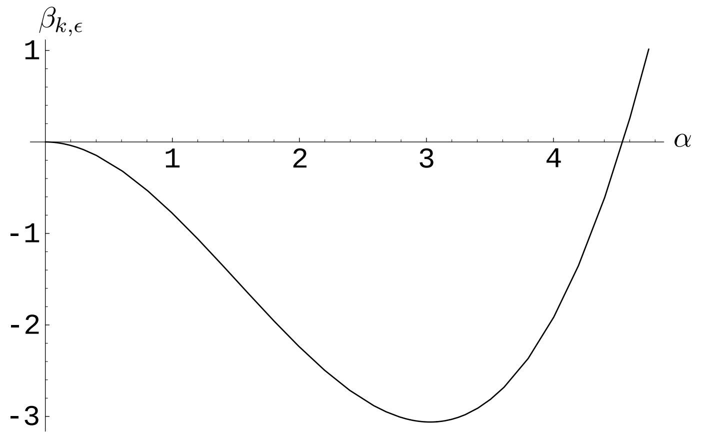

Figure 4: The boundary condition curve $\beta_{k,\epsilon}(\alpha) = -k\alpha^2 + \epsilon\alpha^3$ with $k = 1$ and $\epsilon = .22$.

curve decreases faster and reaches larger (negative) values of β. Given a choice of boundary conditions $\beta(\alpha)$, the allowed black hole solutions are simply given by the points where the black hole curves intersect the boundary condition curve: $\beta_{R_e}(\alpha) = \beta(\alpha)$.

From the black hole curves shown in figure 3, it follows immediately that with AdS-invariant boundary conditions $\beta_k = -k\alpha^2$, there is precisely one hairy black hole solution for each radius R_e, as was discussed in [17]. By contrast, the curve $\beta_{k,\epsilon}$ shown in figure 4, which is defined by the modified boundary conditions (4.3), intersects the black hole curves twice for small R_e. On the other hand, it is clear there are no large hairy black holes that obey the $\beta_{k,\epsilon}$ boundary conditions. The size of the largest hairy black hole solution increases for decreasing ϵ.

The small α (and hence small ϕ_e) black holes form one branch of solutions, which also exist in the theory with $\beta_k = -k\alpha^2$ boundary conditions. Their mass is positive, as shown in figure 5 with $k = 1$ and $\epsilon = .22$, and in fact they are always more massive than a Schwarschild-AdS black hole of the same size [17]. On the other hand, the solutions with the larger α, which are associated with the second intersection point, have much smaller mass. In fact, provided ϵ is sufficiently small, some hairy black holes on this second branch have negative mass, as we illustrate in figure 5. There is thus a black hole with scalar hair on the second branch with the same mass as our initial data. This is the natural endstate of the evolution of the initial data defined by the instanton.[9]

Support for this comes from the fact that as ϵ decreases, the second branch of black holes moves down to lower mass. Hence, the size of the black hole with mass equal to our initial data increases. In the limit $\epsilon \to 0$, the black hole becomes infinitely large. Furthermore, $\phi_e \to \phi(0)$, where $\phi(0)$ is the value of the scalar field on the lightcone that emanates from the origin of the instanton initial data. Therefore, the bulk evolution with boundary conditions defined by the modified field theory with a stable ground state does not describe the formation of a cosmological singularity, but rather a large black hole with

[9]The instanton initial data that obey $\beta_{k,\epsilon}(\alpha)$ boundary conditions with $k = 1$ and $\epsilon = .22$ have mass $M/4\pi = -.057$.

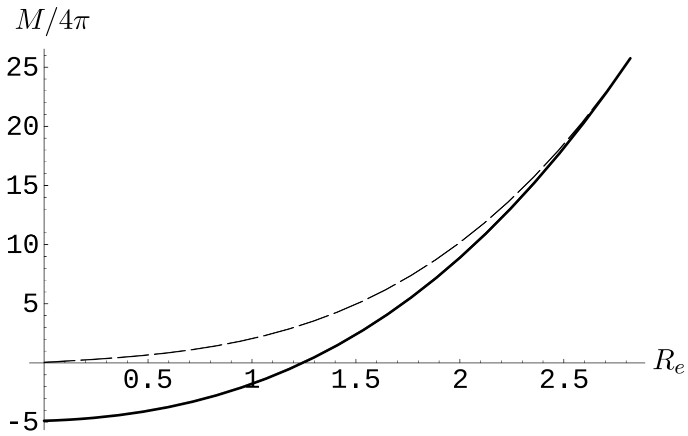

Figure 5: The mass of the hairy black holes that obey the boundary conditions $\beta_{k,\epsilon}(\alpha)$ with $k = 1$ and $\epsilon = .22$. The full line gives the masses of the second branch of solutions, which are associated with the second intersection point of the curves $\beta_{k,\epsilon}(\alpha)$ and $\beta_{R_e}(\alpha)$, and hence have more hair. This branch disappears in the limit $\epsilon \to 0$.

scalar hair. As one removes the regulator ϵ, the minimum of the potential approaches minus infinity and the size of the black hole diverges.

4.3 Dual field theory description of hairy black holes

The hairy black holes found above have a natural description in terms of the dual field theory which we now describe. It was shown in [18] how to compute the effective potential for the vacuum expectation value of the dual operator $\mathcal{O}$ in presence of arbitrary deformations $W(\mathcal{O})$. If S_0 is the action of the usual 2+1 CFT that is dual to M theory with $\beta = 0$ boundary conditions, and

$$S = S_0 + \int W(\mathcal{O}) \tag{4.10}$$

then the expectation values of $\mathcal{O}$ in different vacua are obtained by finding nonsingular bulk solitons with boundary conditions $\beta = W'$ [21, 23]. Given a soliton with $\beta = W'$, one has $\langle \mathcal{O} \rangle = \alpha$. To find the effective potential, one starts with the curve $\beta_s(\alpha)$ obtained from the regular soliton solutions and shown in figure 3. One then defines a function

$$W_0(\alpha) = -\int_0^\alpha \beta_s(\tilde{\alpha}) d\tilde{\alpha} \tag{4.11}$$

and sets

$$\mathcal{V} = W_0 + W\,. \tag{4.12}$$

It follows immediately that the extrema of $\mathcal{V}$ are in one-to-one correspondence with solitons that obey the boundary conditions $\beta = W'(\alpha)$. So the location of the extrema yield $\langle \mathcal{O} \rangle$. Furthermore, one can show that the value of $\mathcal{V}$ at the extremum gives the energy of the corresponding soliton and hence also the energy of the dual field theory state [18]. Therefore one can interpret $\mathcal{V}$ as the effective potential for $\langle \mathcal{O} \rangle$. The fact that the function W does not receive any corrections in the effective potential is reminiscent of a nonrenormalization theorem.

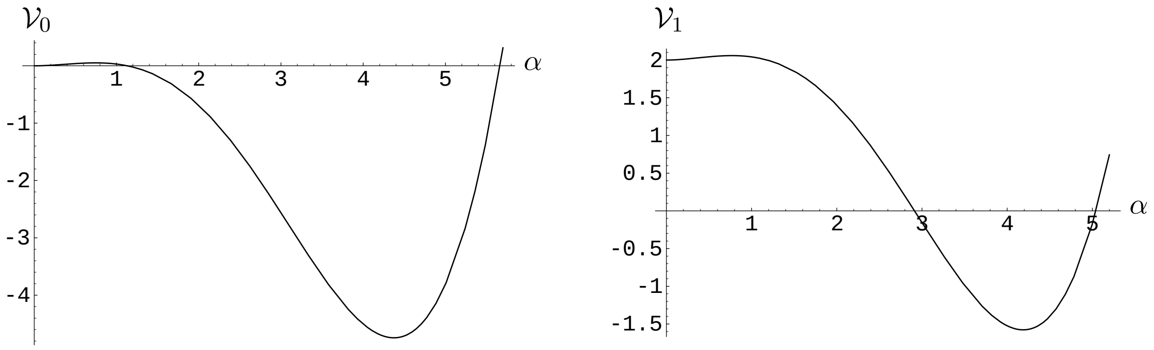

Figure 6: The left panel shows the effective potential $\mathcal{V}_0$ for the vacuum expectation values $\langle\mathcal{O}\rangle$ in the field theory with deformation $W = -\frac{1}{3}\alpha^3 + .055\alpha^4$. The right panel shows the effective potential $\mathcal{V}_1$ for $\langle\mathcal{O}\rangle$ in the same theory, but with the expectation value taken in finite energy states dual to different black holes of size $R_e = 1$.

The left panel of figure 6 shows the effective potential in the deformed field theory that we discussed earlier,

$$W(\mathcal{O}) = -\frac{1}{3}\mathcal{O}^3 + \frac{\epsilon}{4}\mathcal{O}^4 \tag{4.13}$$

with $\epsilon = .22$ which corresponds to the modified boundary conditions $\beta_{k,\epsilon}$ in the bulk. Note that $\mathcal{V}$ has three extrema: a local minimum at $\alpha = 0$ and two extrema with $\alpha \neq 0$. So this field theory has three different vacua and one can consider excitations about each. The usual Schwarzschild-AdS black holes correspond to a typical excitation of mass M about the $\alpha = 0$ vacuum.[10] The top branch of the hairy black holes in figure 5 corresponds to excitations about the local maximum of $\mathcal{V}$, and the bottom branch corresponds to excitations about the global minimum. This interpretation is motivated by several facts. First, for a given size black hole, the top branch is clearly more massive than the bottom. Second, in the limit $\epsilon \to 0$ where one removes the global minimum, the bottom branch of hairy black holes is absent. The top branch is essentially unchanged (although it now continues for arbitrarily large R_e). This is consistent with the fact that the local maximum of the potential is essentially unchanged in this limit. Finally, and most importantly, the upper branch of black holes is unstable. We have shown that there is an unstable spherically symmetric scalar perturbation. This instability is directly analogous to the instability found for hairy black holes with $\epsilon = 0$ boundary conditions [33]. In constrast, this unstable mode is not present for the hairy black holes in the lower branch, and we believe they are stable. This agrees with the stability of the corresponding field theory vacua.

Further information about the black hole states in the dual field theory can be obtained by considering the expectation value of $\mathcal{O}$ in the black hole state. We can define an effective potential whose extrema are precisely these expectation values by generalizing our earlier discussion of the vacuum expectation values. Recall that by considering all hairy black holes with horizon radius R_e, one obtains a curve $\beta_{R_e}(\alpha)$ (see figure 3). For each radius

[10]This applies to most black holes, but not the very small ones. For small excitation energy, the typical state corresponds to a thermal gas surrounding the soliton.

we define a function

$$W_{R_e}(\alpha) = R_e(1+R_e^2) - \int_0^\alpha \beta_{R_e}(\tilde{\alpha})d\tilde{\alpha}\,. \tag{4.14}$$

This function is universal, in the sense that it is independent of the choice of boundary conditions. For any $W(\alpha)$, the function

$$\mathcal{V}_{R_e}(\alpha) = W_{R_e}(\alpha) + W(\alpha)\,, \tag{4.15}$$

clearly has extrema precisely when there are black holes of size R_e that obey the boundary conditions $\beta = W'(\alpha)$. So the location of the extrema give the expectation value of $\mathcal{O}$ in the black hole state. In addition, we now show that the the value of $\mathcal{V}_{R_e}$ at each extrema gives the mass of the corresponding black hole and hence the energy of the field theory state. Suppose we choose our boundary condition to be $\beta = \beta_{R_e}(\alpha)$. In this case, all hairy black holes of size R_e are allowed by the boundary conditions.[11] Furthermore, since the area is fixed, the first law of black hole mechanics implies $dM = TdS = 0$. In other words, all black holes have the same mass in this theory, which equals the mass of the Schwarschild-AdS black hole with radius R_e (which is the $\alpha = \beta = 0$ point on the curve) . Setting $M = 4\pi R_e(1+R_e^2)$ and using (2.11) it follows that

$$W_{R_e} = M_0 + \alpha\beta\,. \tag{4.16}$$

Hence for general boundary conditions $\beta = W'(\alpha)$ one obtains

$$M = 4\pi(W_{R_e} + W) = \oint \mathcal{V}_{R_e} d\Omega \tag{4.17}$$

where we have used the fact that $\beta = \beta_{R_e}(\alpha)$ for all black hole solutions. Thus for general boundary conditions $\beta(\alpha)$ the mass of the black holes (including Schwarschild-AdS) with these boundary conditions is given by the value of $\mathcal{V}_{R_e}$ at the corresponding extrema. Hence $\mathcal{V}_{R_e}$ is an effective potential for the expectation value of $\mathcal{O}$ in the black hole state.

In the right panel of figure 6 we plot the effective potential $\mathcal{V}_1$, constructed from the $R_e = 1$ curve, in the deformed field theory (4.13). One sees that like the vacuum effective potential, $\mathcal{V}_1$ has a local minimum at $\alpha = 0$ and two additional extrema at $\alpha = \langle\mathcal{O}\rangle \neq 0$. The value $\mathcal{V}_1(0)$ equals the mass of Schwarschild-AdS with $R_e = 1$, while $\mathcal{V}_1$ at its local maximum/global minimum gives the masses of respectively the $R_e = 1$ hairy black hole on the upper/lower branch in figure 5. Note that the distance $\Delta\alpha$ between both $\alpha \neq 0$ extrema as well as the difference $\Delta\mathcal{V}$ between their energies are smaller in $\mathcal{V}_1$ compared to $\mathcal{V}_0$. These differences further decrease for increasing mass. For sufficiently massive hairy black holes the extrema of the corresponding effective potential eventually merge and then disappear, which in the bulk corresponds to the fact that there is a maximum size hairy black hole (where the two branches in figure 5 meet).

4.4 Finite temperature field theory

In the context of the AdS/CFT correspondence, rather than working with states of fixed energy as above, black holes are often discussed in the context of a canonical ensemble, in

[11] In fact, there are no black holes of any other size in this theory.

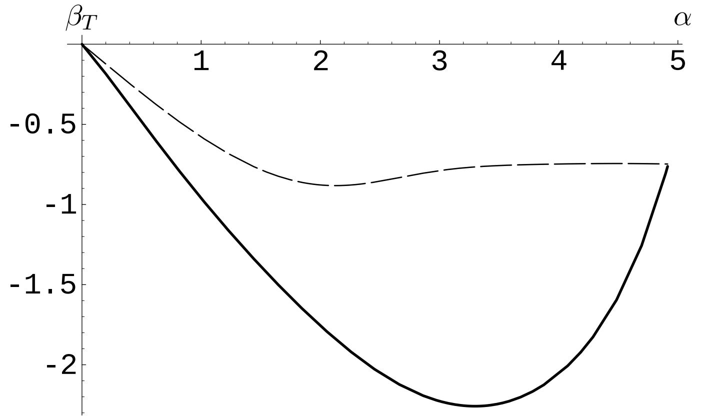

Figure 7: The constant temperature curve $\beta_T(\alpha)$ for $4\pi T = 7$.

which they are dual to thermal states in the field theory. We now show how to compute a finite temperature effective potential in the dual field theory.

The temperature of a hairy black hole is given by

$$4\pi T = \frac{1 - V(\phi_e)R_e^2}{R_e} e^{\delta_\infty} \tag{4.18}$$

where

$$\delta_\infty = -\frac{1}{2}\int_{R_e}^{\infty} r\phi_{,r}^2 dr\,. \tag{4.19}$$

One sees that, like Schwarschild-AdS black holes, hairy black holes only contribute to the thermodynamic ensemble at sufficiently high temperatures, $2\pi T \geq \sqrt{-V(\phi_e)}e^{\delta_\infty}$. By adjusting R_e and ϕ_e so that the temperature (4.18) is held fixed, one finds a one-parameter family of hairy black holes with the same temperature T. From the asymptotic value of the scalar field, we obtain a curve $\beta_T(\alpha)$ shown in figure 7 for $4\pi T = 7$. This curve consists of two branches that are smoothly connected at the maximum value of α. However, the upper/lower branch tend to a *different* Schwarschild-AdS solution at the origin of the (α, β)-plane, namely the usual small/large black hole with $4\pi T = 7$. In general, the upper branch, which tends to the soliton curve for $T \to \infty$, represents the smaller black holes associated with a given temperature, while the lower branch contains the larger black holes (we note, however, that the radius R_e is not constant along the curves $\beta_T(\alpha)$). Thermal states in the field theory are dual to the larger black hole, so we focus on the lower branch of this curve. As T increases, this lower branch moves down and the maximum value of α increases, so the area enclosed by the two constant temperature curves increases too.

The lower branch $\beta_T(\alpha)$ can be used to define a function

$$W_T(\alpha) = \frac{\bar{F}_T}{4\pi} - \int_0^{\alpha} \beta_T(\tilde{\alpha})d\tilde{\alpha}\,, \tag{4.20}$$

where the constant $\bar{F}_T$ is the free energy of the larger Schwarschild-AdS black hole of temperature T. As before, for any $W(\alpha)$, the extrema of

$$\mathcal{F}_T(\alpha) = W_T(\alpha) + W(\alpha) \tag{4.21}$$

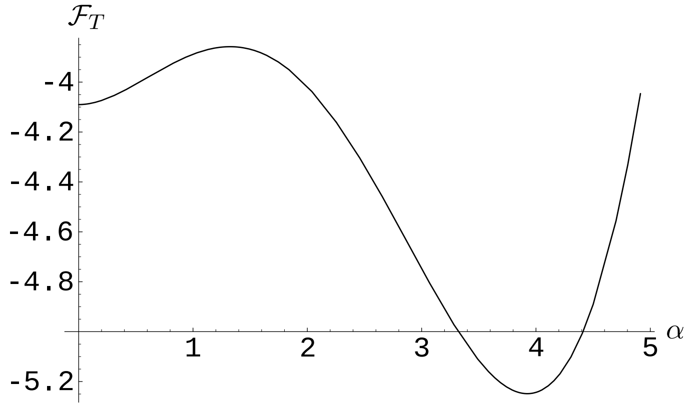

Figure 8: The effective potential for the free energy $\mathcal{F}_T = W_T - \frac{1}{3}\alpha^3 + .055\alpha^4$ at temperature $4\pi T = 7$, constructed from the lower branch of the $\beta_T(\alpha)$ curve.

correspond precisely to hairy black holes of temperature T that obey the boundary condition $\beta = W'(\alpha)$. In addition, we now show that the value of $\mathcal{F}_T$ at each extremum gives the free energy of the corresponding hairy black hole and hence also the dual field theory configuration. Suppose the boundary conditions were $\beta = \beta_T(\alpha)$. Then all the black holes along the lower branch constant T curve are allowed. The first law $dM = TdS$ now implies that the free energy $M - TS$ is constant along this curve.[12] The value of this constant is fixed by the $\alpha \to 0$ limit to be the Schwarzschild-AdS value so

$$W_T(\alpha) + \frac{TS(\alpha)}{4\pi} = M_0 + \alpha\beta\,. \tag{4.22}$$

Hence for general boundary conditions $\beta = W'(\alpha)$ we have

$$F_T = 4\pi(W_T + W) = \oint \mathcal{F}_T d\Omega\,. \tag{4.23}$$

Therefore in the dual field theory one can interpret $\mathcal{F}_T$ as the finite temperature effective potential. We illustrate this in figure 8, where we plot the effective potential $\mathcal{F}_T$ for $4\pi T = 7$ in the field theory dual to $\beta_{k,\epsilon}$ boundary conditions. We see that the hairy black holes that are thermal states in the global minimum are thermodynamically favoured.

4.5 The big bang as a rare fluctuation

We now return to the dual description of our AdS cosmologies. We have seen that a modification of the boundary conditions in the bulk, corresponding to the regularization of the dual field theory, turns the big crunch into a giant black hole with scalar hair. Since

[12]Since the upper and lower curves in figure 7 join smoothly at a maximum α, one might expect the free energy to be constant along the entire curve. However this would contradict the fact that the two Schwarzschild-AdS black holes of the same temperature have different free energies. The resolution is that one must fix the boundary conditions. The free energy is constant only along a continuous family of black holes with the same temperature in the same theory. Since the boundary conditions must be a single valued function of α, the upper and lower branch cannot both be present in one theory.

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the mass of this black hole is negative and much larger black holes exist with these same boundary conditions (see figure 5) we must ask if the black hole we form corresponds to the larger or smaller one of the given temperature. It is easy to see that for small ϵ we always form the larger black hole. This is because in this regime, ϕ_e remains bounded and R_e is very large, so that the temperature of our black hole is very high. The other black hole of the same temperature would instead have to be very small. Hence the black hole we form indeed corresponds to a thermal state in the field theory. As expected, the formation of the large hairy black hole in the bulk corresponds on the field theory side to the zero mode rolling down the potential, exciting all the inhomogeneous modes, and eventually producing a thermal state in the new global minimum that arises from the regularization.

Notice that the evolution in the bulk is nearly independent of ϵ for while. In particular, the light cone of the origin of the (time symmetric) initial data, expands out to large radius in all cases. If ϵ is nonzero, it eventually stops expanding and becomes the event horizon, while if $\epsilon = 0$, it continues to expand and reaches infinity at the same time as the big crunch. This shows that the approach to the big crunch is identical to the formation of a large black hole. In the dual field theory, this is the statement that when the field rolls down the potential, it does not know if there will be a global minimum or not.

If one introduces a large radius cut-off in the bulk, one cannot tell the difference between a very large black hole and a big crunch. But this IR cut-off in the bulk corresponds to a UV cut-off in the dual field theory. This suggests that the evolution to a big crunch can also be viewed as evolving to an equilibrium state in the dual theory. One can view this equilibrium state in the bulk as having all the Planck scale degrees of freedom excited. It would not correspond to any semiclassical spacetime.

For the field to roll back up the potential would require an exceedingly rare fluctuation, which converts all the energy of the approximately thermal state back into the homogeneous mode. In the bulk this would correspond to the time reversed evolution in which there is a big bang singularity in our past. This would mean that the big bang in our AdS cosmologies should be viewed as a rare fluctuation from a generic equilibrium state in quantum gravity. One can imagine that the boundary theory spends a long time in this equilibrium state, which does not describe any semiclassical spacetime. A semiclassical spacetime only arises after a rare fluctuation which, in the dual field theory, corresponds to most of the energy going back into the zero mode.

This picture leads to a natural asymmetry between past and future singularities: the evolution from a past singularity requires a rare fluctuation which, in our context, causes the zero mode to shoot up the potential. In contrast, the approach to a big crunch is the generic evolution corresponding to the field rolling down the potential.

It is natural to speculate that this asymmetry is more general. If so, this could help explain the origin of the second law of thermodynamics in realistic cosmologies. Penrose has stressed that our universe started in an extremely low entropy state,[13] and suggested that this was a result of a fundamental difference between past and future singularities [10].

[13]His estimate for the maximum possible entropy comes from putting all the matter in the observable universe into a large black hole. This is similar to the large black hole in our regularized theory.

The picture we are led to here is similar. However, while Penrose has argued that one needs a time asymmetry in the laws of nature to explain this difference, this does not seem to be present in AdS/CFT.

Clearly, the fact that a rare fluctuation is required for a semiclassical spacetime is not sufficient to explain the second law, since the universe today is semiclassical and has much larger entropy than the early universe. To make further progress one needs a better understanding of the quantum gravitational transition that describes the emergence of semiclassical spacetime from the generic quantum gravity state we envision. This would allow one to compute the relative probabilities of different universes.

If a semiclassical spacetime is the result of a rare fluctuation, then the absolute probability for any feature of the observed universe would be exceedingly small. Of more interest would be conditional probabilities, in which one assumes some gross feature of the universe is present and then asks about the probability for other features [34]. One could, for instance, ask whether an expanding universe like ours is likely to have an early period of inflation. At present the literature contains conflicting statements on this issue.

Dyson et. al. [35] have argued that if the big bang is a rare fluctuation then it would be more likely for a universe like ours to depend on "statistical miracles", instead of evolving in a way that can be understood by usual physical reasoning. However, their starting point was a low temperature thermal state in de Sitter space. De Sitter holography (the idea that one causal patch includes all the degrees of freedom) then implied that a fluctuation that produced the big bang and an inflationary universe must include all the degrees of freedom, which made such fluctuation extremely unlikely.

On the other hand, Albrecht et al. [36] have performed an alternative calculation, in which they take the same starting point but calculate the relative probabilities from more traditional semiclassical tunneling rates (instead of invoking a principle of causal patch physics). In contrast with [35], they find that inflation is strongly favored over other paths to our observed universe.

Finally, Hartle and Hawking have put forward a definite proposal for the wave function of the universe [1]. This provides the most concrete framework to date to compute probabilities for different semiclassical spacetimes. One finds again that the observed universe is more likely to have an early inflationary phase than to arise from a fluctuation directly in its present state [37]. Furthermore, one can also compare the relative probabilities of different inflationary histories. In theories where the inflaton potential has a maximum, for example, one finds a universe like ours is most likely to emerge in a de Sitter state via the Hawking-Moss instanton with a homogeneous field at the maximum [34].

The Hartle-Hawking wave function is peaked around semiclassical geometries. It is therefore likely to differ from the equilibrium quantum gravity state[14] we propose, in which the a priori probability for semiclassical spacetime would be exceedingly small. However, the predictions of probabilities *conditioned* on there being semiclassical spacetime may well turn out to be in good agreement.[15] It would be interesting to see if further work on the

[14]The generic equilibrium quantum gravity state we envision can be viewed as a wave function that defines initial condition for cosmology.

[15]Possible connections between the Hartle-Hawking wave function and string theory were also discussed

AdS cosmologies yields a more concrete understanding of the wave function. This could potentially place one of the existing proposals for the initial conditions in cosmology on firmer footing. A few directions to obtain a more complete understanding of the quantum description of AdS cosmology are outlined in the next section.

5. Discussion

We have seen that the dual description of an AdS cosmology involves a field theory with a potential that is unbounded from below. We have used various approaches to study its dynamics. In a semiclassical analysis, the evolution ends in finite time. In a full quantization of the homogeneous mode, evolution continues for all time and suggests a bounce. However this is an artefact of throwing away all the inhomogeneous degrees of freedom. If one regulates the potential so that it has a global minimum, one sees all the inhomogeneous modes become excited and the system evolves into a thermal state about the true vacuum. However, the corresponding bulk evolution now produces a large black hole with scalar hair rather than a cosmological singularity. The cosmological singularity arises as the limit of a specific class of hairy black holes as the regulator ϵ is taken to zero. This suggests that the approach to a big crunch is naturally viewed in the dual theory as a similar evolution to an equilibrium state, but one which does not describe a semiclassical spacetime.

We also saw in section 4 how one can use hairy black holes in the bulk to compute various effective potentials in the dual field theory. Another application of this discussion is to designer gravity [18]. We have recently shown that one can "pre-order" solitons in the theory (2.1) in the sense that for *any* function $\mathcal{V}(\alpha)$ with $\mathcal{V}(0) = 0$, there are boundary conditions such that the gravitational theory has solitons precisely at the extrema of $\mathcal{V}$ with mass given by the value of $\mathcal{V}$ at the extrema. It is easy to show that one can pre-order hairy black holes as well, either in terms of their radius or temperature. For example, suppose one wants to specify the mass of hairy black holes of radius R_e in the theory (2.1). Given any function $\mathcal{V}_{R_e}(\alpha)$ with $\mathcal{V}_{R_e}(0)$ equal to the Schwarzschild-AdS mass, one defines $W(\alpha)$ via eq. (4.15) and chooses boundary conditions $\beta = W'$. It follows from our earlier discussion that the resulting theory has hairy black holes of this radius at each extremum of $\mathcal{V}_{R_e}(\alpha)$ and the mass of the black hole will be given by the value of this function at its extrema.

The dual description of AdS cosmologies suggests that the big bang and the emergence of semiclassical spacetime is an exceedingly rare fluctuation from a generic equilibrium state in quantum gravity. A few possible directions to obtain a more complete quantum description of AdS cosmologies are the following. Clearly one needs a better understanding of the equilibrium field theory state when the regulator ϵ is taken to zero. Since we are driven to arbitrarily large values of the fields in this limit, one might wonder if the AdS/CFT correspondence breaks down, and whether the dual field theory must be extended to include more stringy (or M-theory) effects.[16] This is suggested by the fact that the original AdS/CFT correspondence arose by taking a low energy limit of the excitations of a stack

in [38] and [39].

[16] We thank D. Gross for suggesting this possibility.

of branes, and we are being driven to consider large excitations. Another interesting direction is to study the BKL chaos in AdS cosmology. It has been shown that in the full 11D supergravity evolution, the generic approach to a cosmological singularity is chaotic with different spatial points decoupling and undergoing a series of Kasner oscillations [2]. It would be interesting to relate this to the dynamics of tachyonic preheating in the dual field theory. This relation will have to include the fact that our truncated theory of four dimensional gravity coupled to a scalar does not exhibit this chaotic behavior [40, 41]. Finally, Banks and Fischler [7] have given a description of the big crunch which has some similarities to ours. In particular, they propose the big crunch is a state which does not correspond to a semiclassical spacetime, maximizes the entropy,[17] and is related to a conformal symmetry. It would be interesting to explore if there is a deeper connection.

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A. Glossary

The functions $\beta(\alpha)$ refer to the asymptotic behavior of the bulk scalar field (2.3). We list here the various functions that are used in the text:

- The function $\beta_i(\alpha)$ is defined by the $O(4)$-invariant instantons. It is shown in figure 1.
- The function $\beta_s(\alpha)$ is defined by the spherical solitons. It is shown in figure 3.
- The functions $\beta_{R_e}(\alpha)$ are defined by the hairy black holes of size R_e. A few examples are shown in figure 3.
- The functions $\beta_T(\alpha)$ are defined by the hairy black holes of temperature T. An example is shown in figure 7.
- The functions $\beta_k(\alpha)$ denote the AdS-invariant boundary conditions $\beta_k(\alpha) = -k\alpha^2$.
- The functions $\beta_{k,\epsilon}(\alpha)$ denote the modified boundary conditions $\beta_{k,\epsilon}(\alpha) = -k\alpha^2 + \epsilon\alpha^3$ (see figure 4).
- Without subscript, the function $\beta(\alpha)$ refers to an arbitrary boundary condition.

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